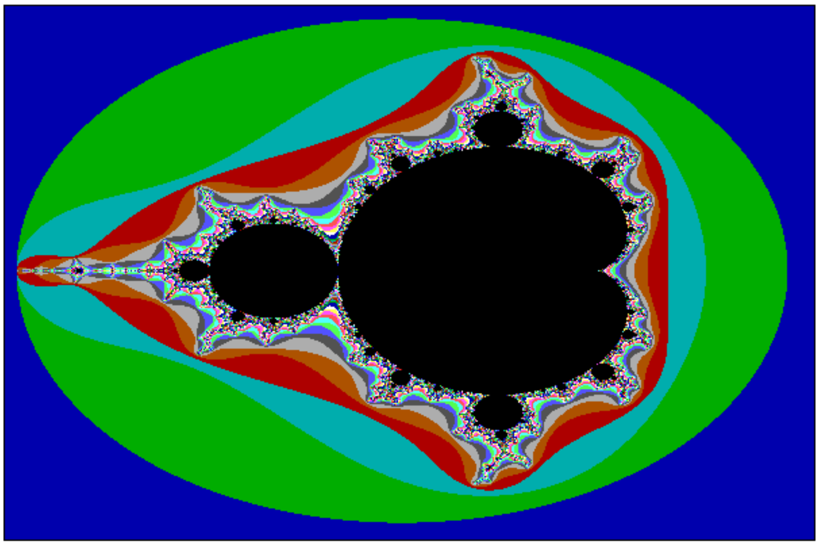
Lecture 14

Fractals Ch 9

Some missing pieces

Evaluation

Most famous fractal: Mandelbrot set



What is it, more than a pretty image?

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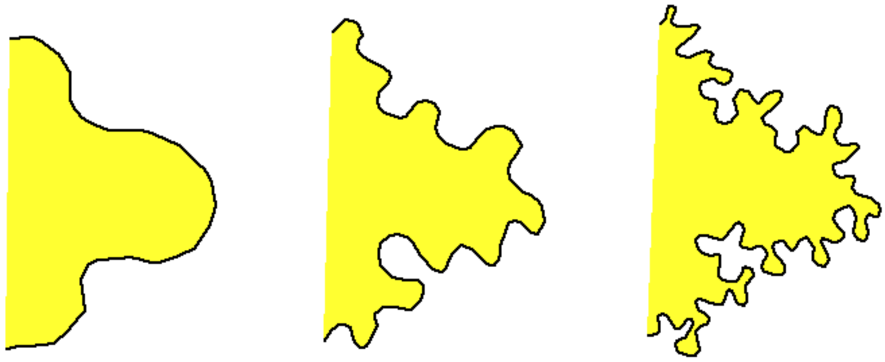
Natural objects have fractal features

Classic example: Coastline

Shape and length varies with resolution



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Fractals in computer graphics

Fractals are shapes with:

- self-similarity
- infinite resolution

Used for modelling such shapes

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Classification of fractals

- geometrical recursive construction
- stochastic fractals
- mathematical formulas (in the complex plane)

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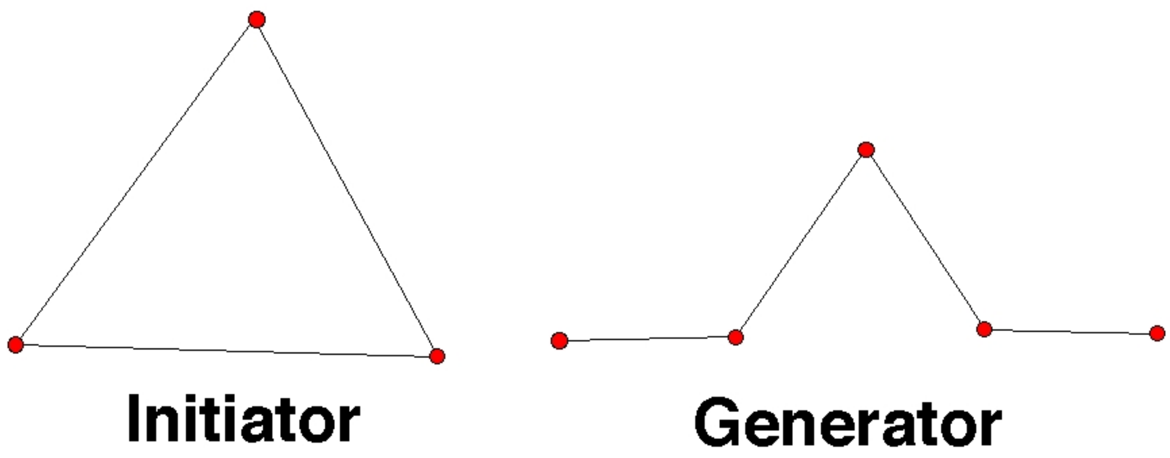
Geometric construction of self-simlar fractals

Example: Koch curve



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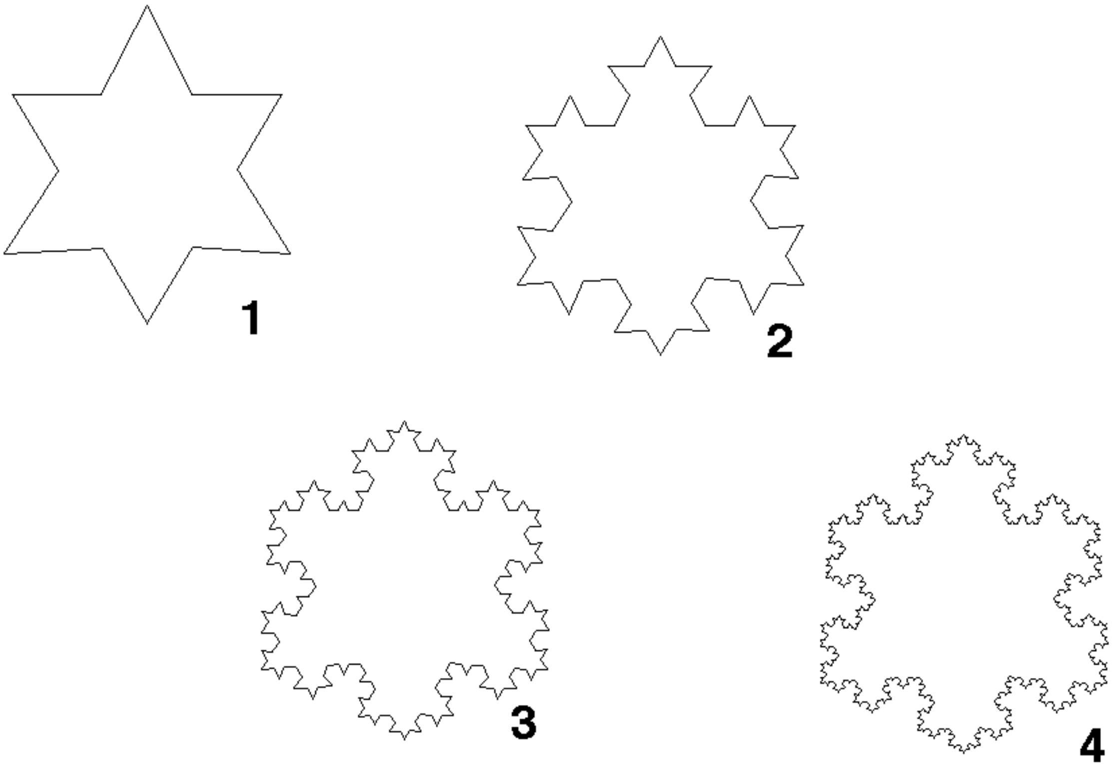
Resulting Koch curves



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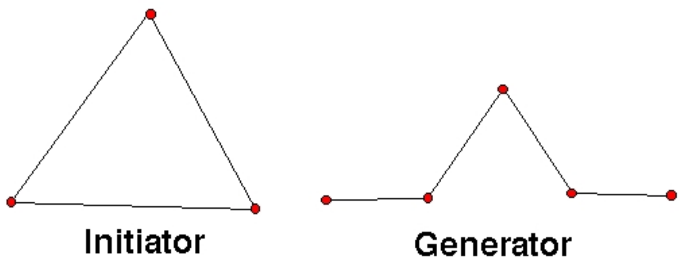

Recursive function

Pass all parts to next level

Replace part with the generator, scaled to same length

Stop at desired recursion depth or when sections are small enough (e.g. 1 pixel long)

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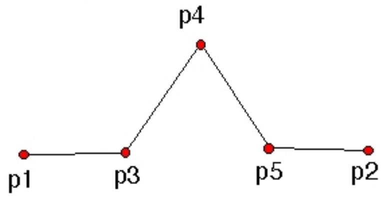
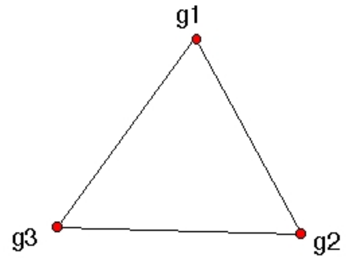
```
procedure DrawKoch(p1, p2, depth)

if depth >= maxDepth then

        MoveTo(p1)
        LineTo(p2)
        return

else

        calculate p3, p4, p5 as the three points inside the generator

        DrawKoch(p1, p3, depth+1)
        DrawKoch(p3, p4, depth+1)
        DrawKoch(p4, p5, depth+1)
        DrawKoch(p5, p2, depth+1)
```




```
main procedure:

Choose three generator points, g1, g2, g3

DrawKoch(g1, g2, 0)
DrawKoch(g2, g3, 0)
DrawKoch(g3, g1, 0)
```



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Fractal dimension

A measure of how rough or fragmented the shape is

Definition:

$$ns^D = 1$$

n = number of subparts

s = scaling

D = fractal dimension

Solves to $D = \ln(n) / \ln(1/s)$

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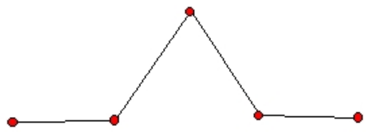
Fractal dimension example:

Koch curve

n = 4

s = 1/3



D = ln 4 / ln 3 = 1.26

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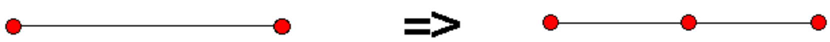
Fractal dimension example:

Splitting a line



n = 2

s = 1/2

D = ln 2 / ln 2 = 1

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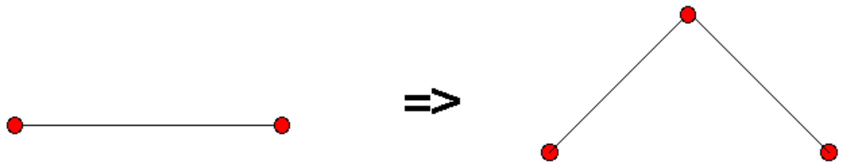
Fractal dimension example:

Splitting a line and moving midpoint



n = 2

s = $1/\sqrt{2}$

$D = \ln 2 / \ln \sqrt{2} = 2$

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Fractal dimension:

In 2D:

1 to 2: Well-behaved fractal curve

>2: Self-intersecting, area-covering

Split line: D = 1 minimum, no fractal
Koch: D = 1.26, moderate fractal
Moved midpoint: D = 2, maximum

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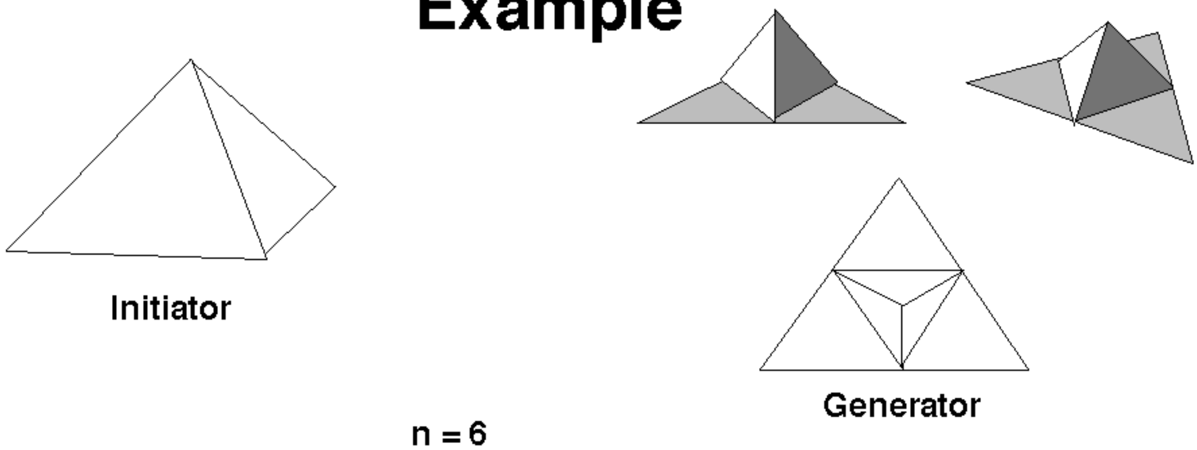
Geometric construction of self-similar fractals in 3D

Example



n = 6

s = 1/2

D = ln 6 / ln 2 = 2.58

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Interpretation of fractal dimension:

In 3D:

2 to 3: Well-behaved fractal surface

>3: Self-intersecting, volume-covering

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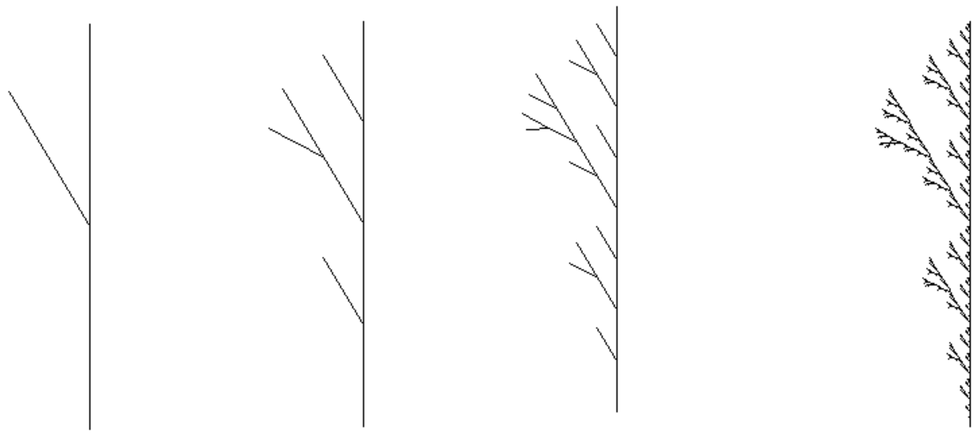
Example: Generation of plants



Promising, but too self-similar!

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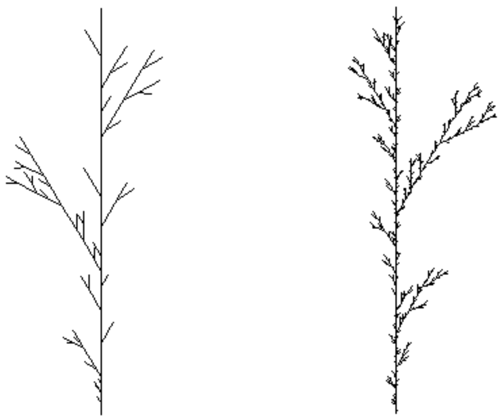
Statistically self-similar fractals

Random variation of generator



Same branch generator as before, with some randomness!

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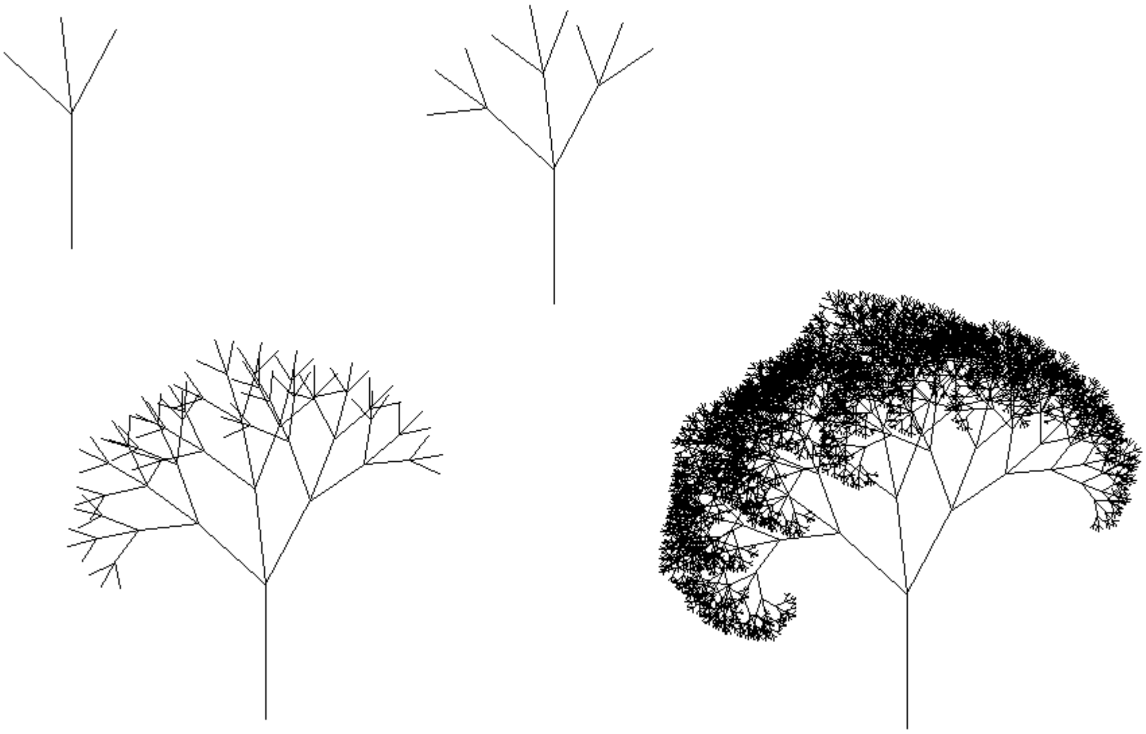
Example: Generation of plants #2



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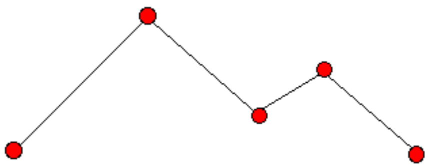
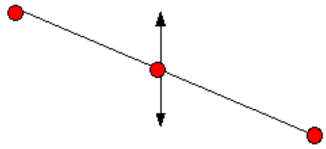
Random midpoint-displacement

Good for fractal terrain generation



Initiator

Desired rough
overall whape



Generator

Find midpoint,
displace along y only

7 iterations

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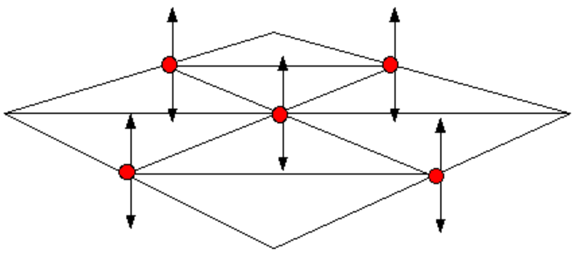
Fractal terrain generation in 3D

Split a square to four

Displace midpoints of each side



Middle point can be independent or calculated from the others

Edge points must match neighbor patches

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Fractal terrain generation in 3D

Heightfield approach

Terrain level k is array of resolution $2^k \times 2^k$

The next level has 4x the resolution

Generate new 2x2 block from one, or filter over a small neighborhood

Add random offset to all values

Offset should be smaller for higher k

=> magnitude of frequency components inverse proportional to frequency!

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Terrain generation by FFT

Related method!

Fill frequency space (2D) with random numbers

Filter by G(f) = F(f) * 1/|f|

Convert to spatial image with FFT

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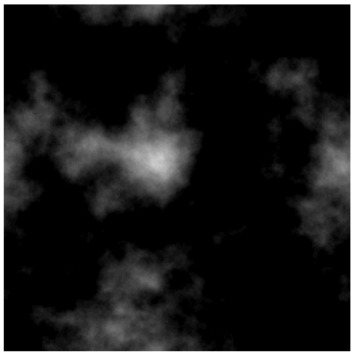
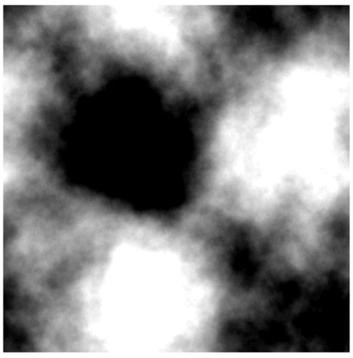
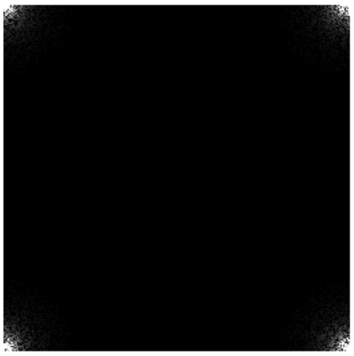
Terrain generation by FFT

Examples





Frequency space:



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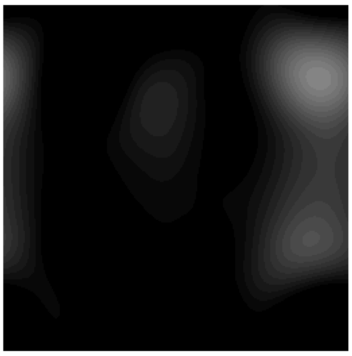
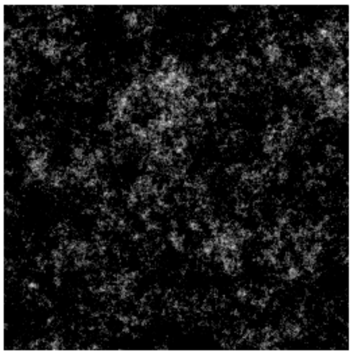
Other falloffs than 1/|f|

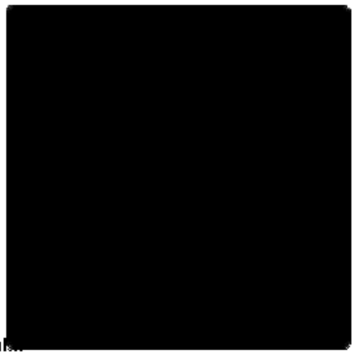
$1/|f|^2$

$1/\text{sqrt}(|f|)$

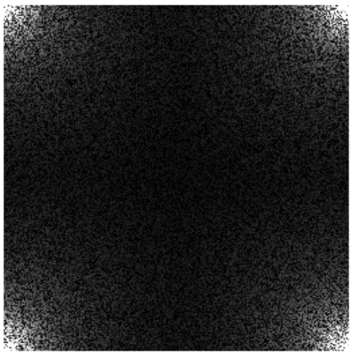


Signal space:





Frequency space:



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Geometric construction of fractals

- Deterministic self-similar fractals
- Statistically self-similar fractals
- Random midpoint displacement

Easy to control

Intuitive parameters

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Self-squaring fractals

Based on simple functions in complex space

Insert complex numbers (points) into a function

Apply function recursively, and analyze the behaviour.

• Diverge?
• Converge?
• Chaotic?

Converge or chaotic: Does it keep within some limit in a number of iterations?

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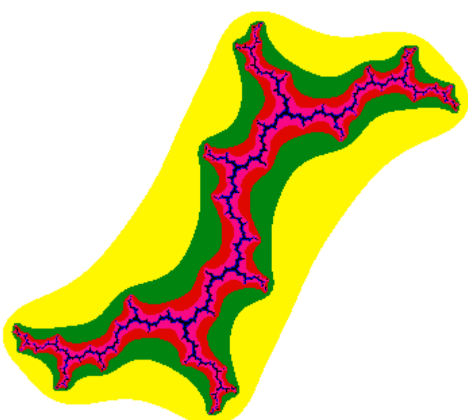
Self-squaring fractals

The Julia set

$$z_{k+1} = z_k^2 + \lambda$$



Julia set for $\lambda = (0, 1) = 0 + j$

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The Julia set

Implementation

```
for y = miny to maxy
    for x = minx to maxx
        (zr, zi) = scaling of (x,y)

        for i = 0 to maxiterations
            z = z² + λ
            if |z| > R then Leave

        Draw pixel (x,y) (different colors for different i)
```

maxiterations ≈ 15

$R^2 \approx 10$

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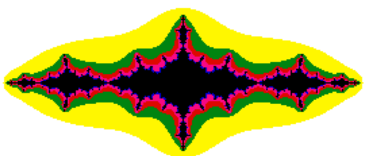
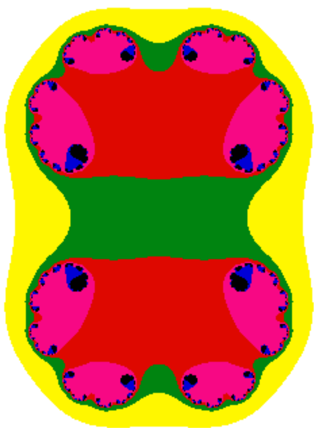
Other Julia sets

$$z_{k+1} = z_k^2 + \lambda$$

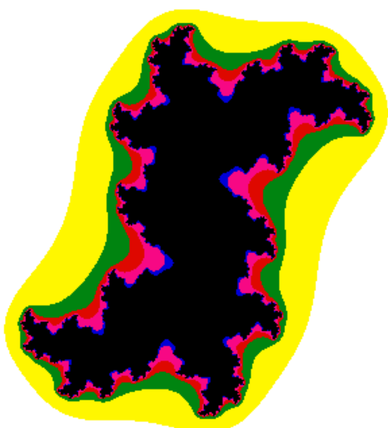
Other λ values



$\lambda = (-1.3, 0)$



$\lambda = (0.4, 0)$



$\lambda = (0.3, 0.5)$

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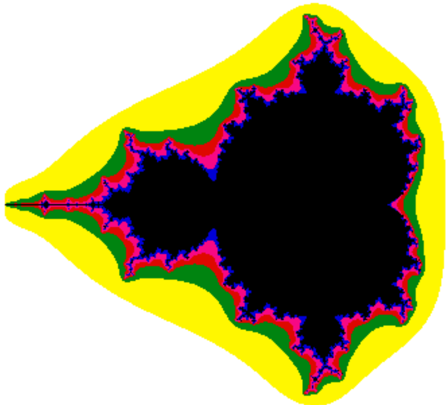
Self-squaring fractals

The Mandelbrot set

$$z_{k+1} = z_k^2 + z_0$$



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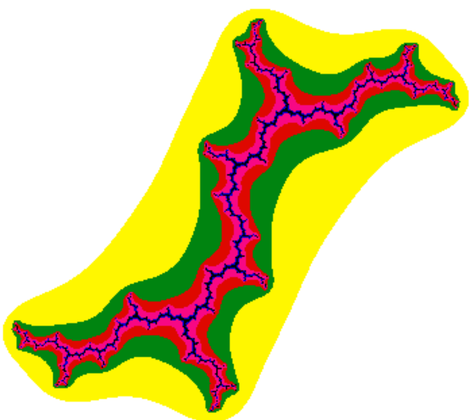
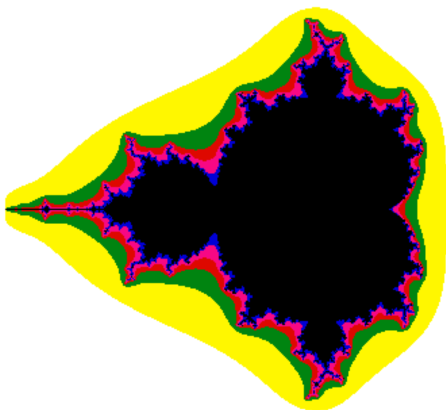
Self-squaring fractals

- Beautiful
- Non-predictable
- Limited usability





Mathematical curiosity

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Fractals, summary

1) Geometrically constructed fractals

Very useful for generating many kinds of natural objects

Allows design of complex models with arbitrary resolution

2) Self-squaring fractals (and other adventures in the complex plane)

Questionable practical usability

Hard to do planned designing

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Related methods:

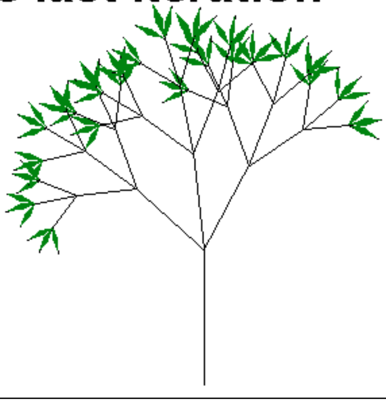
Shape grammars and procedural methods

No unlimited resolution

Different rules at different levels

Example: Tree with leaves: replace last iteration with leaf generator

"graftals"



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